

Network complexity and topological phase transitions

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A new type of collective excitations, due exclusively to the topology of a complex random network that can be characterized by a fractal dimension D_F , is investigated. We show analytically that these excitations generate phase transitions due to the non-periodic topology of the $D_F > 1$ complex network. An Ising system, with long range interactions over such a network, is studied in detail to support the claim. The analytic treatment is possible because the evaluation of the partition function can be decomposed into closed factor loops, in spite of the architectural complexity. This way we compute the magnetization distribution, magnetization loops, and the two point correlation function; and relate them to the network topology. In summary, the removal of the infrared divergences leads to an unconventional phase transition, where spin correlations are robust against thermal fluctuations.

Phase transitions have attracted significant attention over many years. In this context the analytic achievement known as the Onsager [1] solution of the two-dimensional (2D) Ising model and the Mermin-Wagner theorem (MWT) [2, 3] have been important landmarks in the development of the field.

Here we investigate a complex network with Ising nearest-neighbor (nn) plus random long-range interactions, and propose a mechanism that provides a local enhancement of the correlation length. We find that, as a consequence of the inclusion of frustrated long-range interactions, our non-periodic system breaks inversion symmetry, due to its fractal nature. Consequently, when the effective fractal dimension $D_F > 1$ acquires non-integer values, incoherent fluctuations are strongly suppressed.

During more than half a century the dimensionality restrictions imposed by the MWT [2–5], which for integer $D < 3$ is responsible for the suppression of correlations by thermal fluctuations, has posed a challenging problem in areas like condensed matter and high energy physics. During the last two decades the strong dependence of the collective behavior of embedded topologies has given impulse to the study of magnetic systems on more intricate architectures. This way complex networks have become a fertile ground in the study these non-ideal systems [6–8], since in spite of their compact structure, the spatial fluctuations in complex networks do give rise to a wide range of critical phenomena, that for $D < 3$ disappear as the fractal dimension of the network approaches an integer value.

There is a considerable number of publications on the subject of phase transitions on complex networks. For example, the ferromagnetic Ising model on Watts-Strogatz networks displays a ferromagnetic transition at finite temperature [9, 10]. This same phenomenon was also obtained by Monte Carlo simulations, for $D = 2$ and 3 on *small-world* networks, by Herrero [11] and by Hong et al. [12]. These results are in agreement with mean field

theory. However, numerical simulations by Aleksiejuk et al. [13], and predictions based on microscopic theory by Dorogovtsev et al. [8] and by Leone et al. [14], confirm that the phenomenology associated with complex networks goes well beyond mean field theory. Moreover, Yi [15, 16] studied fluctuations of the Ising model on a scale free network including an external field orthogonal to the easy axis [17]. The critical exponent of this critical point has been investigated for several models, such as the Ising-XY in a transverse field [18, 19], and an infinite Ising chain [20, 21].

We note that in these systems the underlying commonality is the emergence of phase transitions produced by the topological structure of the complex network. Hence, in this manuscript we show, mainly using analytic tools, that due to the topology of a (*small-world*) complex network, the local correlation length can increase due to randomly frustrated interactions. Indeed, when the fractal dimension $D_F > 1$ the complexity of the network can yield “long range” correlations of the collective excitations.

We demonstrate the effect by means of two approaches: i) we use a qualitative description that suggests how the topological complexity of the network can produce such a phase transition. This approach, in essence, generalizes Landau’s phenomenological proposal for critical phenomena as they occur in complex networks; ii) we construct an Ising model with the spins at the nodes of the complex network, and they interact through nn and long range interactions. The network in this specific case is of a *small-world* type, which has implications that we will discuss below, for the correlation length and the nature of the phase transition itself [22]. The partition function of the model is calculated analytically, so that all the thermodynamic properties can be evaluated. Below we show, that this phase transition of the Ising model on such a network is robust against thermal fluctuations, and consequently that an unconventional sec-

ond order phase transition does occur. Our theory also reproduces qualitatively the results of Chen et al. [23] that confirmed, through simulations of a 2D eight-state Potts model, that the presence of randomly distributed ferromagnetic bonds changes the phase transition from first to second order. The same analysis can be applied to results that were obtained by Theodorakis et al. [24] with a 2D Blume-Capel model embedded in a triangular lattice, and by Fytas et al. [25] for an antiferromagnetic Ising model with next nearest neighbor interactions.

The concept of “long range” correlations to induce phase transitions requires some discussion. In general, we expect to observe a phase transition, and its associated critical phenomena, when the correlation length becomes large, e.g., effectively of the size of the system. In the case of the Ising model over a complex network of the *small-world* type, it is also true that the correlation length is about the size of the system. However, in this case the complex topology reduces the effective distance between nodes, allowing the system to become of the size of the correlation length. This is why we have used a *small-world* type network, which allows a large system to have small average correlation length. This observation should be quite general, and be applicable to a number of other systems, which we expect on the basis of the phenomenological approach we present below.

We start our analysis with a qualitative characterization of the general phase transition problem in complex networks, particularly of the *small-world* type. The phase a system adopts, in thermal equilibrium, is characterized by spontaneous symmetry breaking [26] and by the ratio of the order parameter and the characteristic length of the system. Our main assumption is that the addition of the *small-world* structure (long range interactions) does not change significantly the correlation length but that, due to cluster decomposition, it reduces its size-scale and consequently increases the local correlation. Following Landau [26] we assume invariance under time reversal inversion and discrete symmetries to write, neglecting higher order terms of the order parameter ψ , the free energy $\Phi_0(\mathbf{x}, T)$ as $\Phi_0(\mathbf{x}, T) = A(T)|\nabla\psi|^2 + B(T)|\psi|^2$, where \mathbf{x} is the spatial coordinate, $\psi = \psi_0 e^{-x/\xi}$, $\xi = \sqrt{A(T)/B(T)}$ is the correlation length, and A and B are functions of the temperature T . For $T > T_c$, $A(T) \ll B(T)$, and consequently $\xi \ll 1$. On the contrary, below the critical temperature $T < T_c$, $A(T) \gg B(T)$, and $\xi \gg 1$. This way the phase transition is characterized analytically. In order to estimate the critical dimensionality, we use the energy equipartition theorem which, including spatial fluctuations of the classical ground state, takes the form

$$\int d^D x \Phi_0(\mathbf{x}, T < T_c) \geq k_B T, \quad (1)$$

where D is the dimensionality of the system and k_B is the Boltzmann constant, and the inequality is due to quan-

tum fluctuations.

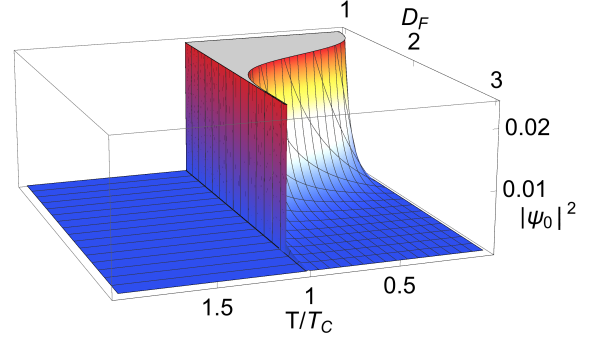


FIG. 1. (color online). Regularization of infrared divergences when the fractal dimension $D_F > 1$.

Using the explicit expression for the order parameter in Eq. (1), we obtain (for $0 < T < T_c$)

$$|\psi_0|^2 \geq \frac{\rho k_B T}{A(T)(\pi\xi)^D \Gamma(D)} \frac{k^{D-2}}{D-2} \Big|_{k \rightarrow 0}^{k_0}, \quad (2)$$

where $1/\rho = \int d^D k / (2\pi)^D$, $\Gamma(y)$ is the Gamma function, and $1/k_0$ is the range of the interaction in units of the nn distance. This leads to a $1/k|_{k \rightarrow 0}$ infrared divergence for $D = 1$, and a $\ln k|_{k \rightarrow 0}$ divergence for $D = 2$, a result that is equivalent to the MWT. It is worth mentioning that our phenomenological model recovers results derived from a microscopic theory, based on the Bogoliubov inequality [27].

Now we focus our attention on how this phenomenological theory can be applied to a complex network with both nn plus long range random links. This system can be characterized in many ways, for example by an effective fractal dimension or an average path length L . The effective fractal dimension D_F can be defined as the scaling $\langle N \rangle \sim d^{D_F}$ of the average number $\langle N \rangle$ of nodes within a radius d of a given node. The average path length L is the average distance $L = \frac{1}{2N(N-1)} \sum_{i \neq j} d_{ij}$, where d_{ij} is the shortest path length between node i and node j , and N is the number of nodes. Therefore, by means of a box-counting procedure, the inversion symmetry breaking may be characterized by an effective fractal dimension, which is expected to couple the free energy to a current density axial vector term, defined as

$$\mathbf{J}(\mathbf{x}) = i(\psi(\mathbf{x})\nabla\psi^*(\mathbf{x}) - \psi^*(\mathbf{x})\nabla\psi(\mathbf{x}))/2. \quad (3)$$

This axial vector $\mathbf{J}(\mathbf{x})$ is the simplest term that one can incorporate to break inversion symmetry. The above equation implies that $|\nabla\psi|^2 \sim J^2/|\psi|^2$, where $J(\mathbf{x}) = |\mathbf{J}(\mathbf{x})|$. This is a higher order term that, for $T < T_c$, can be neglected compared to $\Phi_1(\mathbf{x}, T) = \gamma(T)J(\mathbf{x})$, where

$\gamma(T) \sim k_B(T_C - T)$. When this last term is included in the free energy, we obtain

$$|\psi_0|^2 \geq \frac{\rho k_B T}{\gamma(T)(\pi\xi)^D \Gamma(D)} \frac{k^{D-1}}{D-1} \Big|_{k \rightarrow 0}^{k_0}. \quad (4)$$

This way the infrared divergences are removed for $D > 1$ since $|\psi_0|^2 \sim k^{D-1}/(D-1)$, so that the restrictions on phase transitions for $D < 3$ are removed. With the insight gained from dimensional regularization of the phenomenological approach, we now show how random exchange interactions lead to long range magnetic order. Our starting point, or unperturbed Hamiltonian, is a 1D periodic system (*i.e.* a ring with only nn interactions), and the random link interactions are added perturbatively *i.e.* $J_0/J \ll 1$, the expansion in momentum eigenstates is valid. But, this random link distribution brings about a dimensionality change, from integer to fractal ($D \rightarrow D_F$), and the breaking of the periodicity of the system (which is essential for the MWT to hold). In Fig. (1) we show the temperature and fractal dimension D_F dependence of the order parameter, for $k_0 \sim 1$ and assuming that the elementary excitation density, of correlation length ξ^{D_F}/ρ , remains constant. It is thus apparent that the introduction of random links removes the infrared divergences, and that phase transitions are in principle now allowed.

We now illustrate the procedure with a ring Ising model, in order to analyze and illustrate the characteristics of this phase transition. As is well known, in the absence of random links, the 1D Ising model obeys the MWT and does not exhibit phase transitions [2] due to the periodicity of the interactions. To the 1D ring Ising model composed of N nodes, with only nn interactions, we add the possibility of long range interactions with other particles on the ring, thus generating a complex network. The Ising Hamiltonian with long range random interactions is given by

$$\mathcal{H} = - \sum_i \left(J \sigma_i \sigma_{i+1} + h \sigma_i + J_0 \sigma_i \sigma_{r_i} \right), \quad (5)$$

where $\sigma_i = \pm 1$; $J > 0$ and J_0 are the exchange constants between nn and long range neighbors, respectively, and $|J_0| < |J|$; g is the gyromagnetic ratio; $h = g\mu_B H_0$; μ_B is the Bohr magneton; and H_0 is the uniform applied magnetic field. For simplicity, we consider a single particle, namely r_i , connected to particle i . Consequently, the first term in Eq. (5) describes the nn exchange, the second the Zeeman interaction, and the third the exchange between particle i and the one located at r_i . The effective fractal dimension, and the average path length, grow linearly with $\ln N$, consistent with a *small-world* topology, which increases the connectivity of the network locally and enhances the collective behavior of the system.

In order to calculate effects due to the random links on the phase transition we implement the transfer matrix

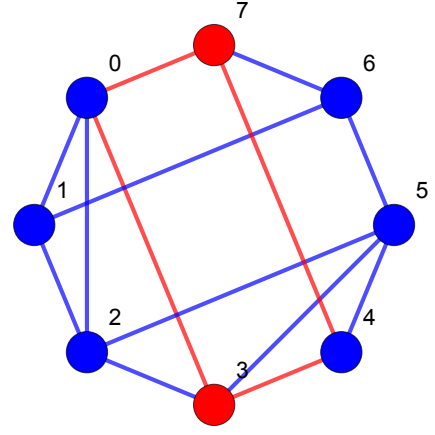


FIG. 2. (color online). Geometric representation of the decomposition into closed factor loops. The loop factors are nodes 064251 (blue) and 37 (red). Following Eqs. (6) and (7) they correspond to the factorization $M = [M_{06} M_{64} M_{42} M_{25} M_{51} M_{10}] [M_{37} M_{73}] = \text{tr}(M^6) \text{tr}(M^2)$

method, to calculate analytically the partition function. We start defining the matrices

$$\begin{aligned} T_{\sigma_i, \sigma_j} &= \langle \sigma_i | e^{\epsilon_{i,j}/k_B T} | \sigma_j \rangle, \\ R_{\sigma_i, \sigma_j} &= \langle \sigma_i | e^{\Delta_{i,j}/k_B T} | \sigma_j \rangle, \\ M_{\sigma_i, \sigma_j} &= \sum_{\sigma_\ell} T_{\sigma_i, \sigma_\ell} R_{\sigma_\ell, \sigma_j}, \end{aligned} \quad (6)$$

where $J_0 \neq 0$, $\epsilon_{i,j} = J \sigma_i \sigma_j + h(\sigma_i + \sigma_j)/2$ and $\Delta_{i,j} = J_0 \sigma_i \sigma_j$. J and J_0 describe the nn and long range interactions, respectively. $\Delta_{i,j}$ describes the interaction between the magnetic moment at sites $i+1$ and r_i through site i . These definitions allow to write the partition function of the system as

$$\begin{aligned} Z &= \sum_{\{\sigma_1, \sigma_2, \dots\}} \sum_{\{\sigma'_1, \sigma'_2, \dots\}} \prod_i T_{\sigma_{i+1}, \sigma'_i} R_{\sigma'_i, \sigma_{r_i}} \\ &= \sum_{\{\sigma_1, \sigma_2, \dots\}} \prod_i M_{\sigma_{i+1}, \sigma_{r_i}} = \prod_k \text{tr}(M^{n_k}) \end{aligned} \quad (7)$$

where n_k is the number of vertices of each closed loop of nodes that is formed as we follow the random links along the network. Eq. (7) specifies the cluster decomposition of the transfer matrix in closed loop factors. This decomposition is illustrated in Fig. (2) for an $N = 8$ configuration.

The $J_0 = 0$ limit is recovered recalling that $Z = \text{tr}(T^N) = \lambda_+^N + \lambda_-^N$, where $\lambda_+ > \lambda_-$ are the T matrix eigenvalues. In the $N \rightarrow \infty$ limit $Z = \lambda_+^N$. However, when long range interactions are incorporated the spatial fluctuations are decomposed into closed loop factors, which in the $N \rightarrow \infty$ limit regularize the infrared divergences because we now have a multiplication of traces which correspond to the closed loop factors, so that

$\text{tr}(M^{n_0}) = \lambda_+^{n_0} + \lambda_-^{n_0}$ must be kept in full, for some $n_0 = \min\{n_k\} \ll N$.

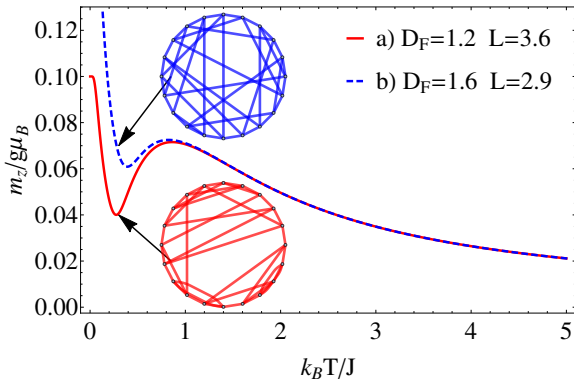


FIG. 3. Magnetization per particle m_z as a function of temperature for two small D_F values, and for two different average path lengths L .

In the absence of an applied field ($h = 0$) and at finite temperatures, the average magnetic moment per particle $m_z = (N\beta)^{-1} \partial \ln Z / \partial h$ vanishes for the Ising model, which implies that long range order is destroyed by thermal fluctuations. However, as the degree of interconnectivity between nodes grows, the fluctuations become suppressed. This in turn leads to an unconventional continuous phase transition for $J_0/J < 0$, *i.e.* when the magnetic configuration is determined by a competition of short and long range interactions of different sign. Due to the local frustration this symmetry breaking could be expected to generate a transition to a spin glass arrangement. However, as a consequence of the *small-world* network topology, below the critical temperature of a locally ordered magnetic structure distribution is favored. This is illustrated in Fig. (3), where we display the magnetization *vs.* temperature for two D_F values, and two different average path lengths. We remark that the dependence of m_z on T has a highly nonlinear dependence on the fractal dimension D_F . This fact is also reflected in the $m_z \neq 0$ values shown in Fig. (4), in the applied magnetic field $h \rightarrow 0$ limit.

In order to describe the effects of the topology on the phase transition we study the correlation length α_K of the K -th closed loop $\mathcal{C}(n_K)$, with n_K vertices, as

$$\begin{aligned} F(n_K) &= \sum_{i, \delta \subseteq \mathcal{C}(n_K)} \langle \sigma_i \sigma_{i+\delta} \rangle / n_K \\ &= \sum_{\delta \subseteq \mathcal{C}(n_K)} \text{tr}(\sigma M^\delta \sigma M^{n_K-\delta}) / \text{tr}(M^{n_K}), \end{aligned} \quad (8)$$

where $F(n_K)$ is the two-point correlation function over a closed loop $\mathcal{C}(n_K)$. On the basis of the above analytic expression, the correlation length α_K can be obtained from

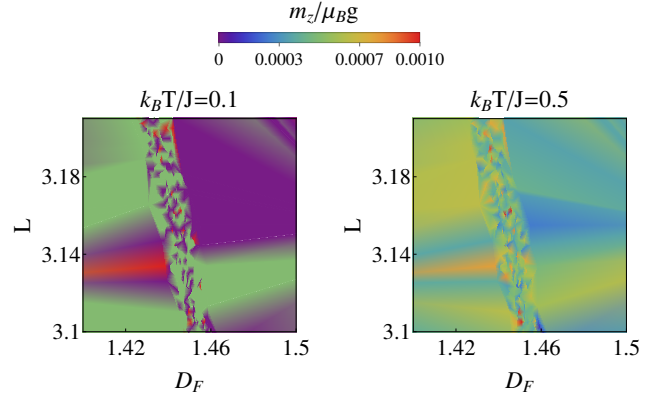


FIG. 4. Magnetization per particle m_z averaged over 10^3 *small-world* network configurations as a function of temperature $k_B T / J$, for $h/J = 10^{-3}$. The color-scale indicates the magnetization magnitude in $g\mu_B$ units. Each one of this *small-world* $N = 20$ node networks corresponds to a 1D Ising ring with one long range link per node.

the exponential dependence [28] of $F(n_K) \propto e^{n_K/\alpha_K}$. The collective behavior of the system is dominated by the local correlation of magnetically ordered clusters. In order to show that, let us consider a 1D Ising ring system with one long range link per node (over $N = 20$ nodes) with a *small-world* network structure. Fig. (5) displays the hysteresis loops and cluster decomposition into local conformations in configuration space. The strong correlation of small clusters becomes quite apparent by inspection of Table (I), which also shows that the magnetic ordering is robust against thermal fluctuations.

As the magnetic field decreases the correlation length is reduced, and the magnetic ordering is destroyed by the thermal fluctuations in large clusters, but it is preserved in small ones. This induces a residual magnetization, where the continuous line represents $J_0 = -|J|/2$ ($J > 0$), and the dashed line the $J_0 = |J|/2$ ($J < 0$) case. It is also worth mentioning that this continuous phase transition resembles ferrimagnetic or canted anti-ferromagnetic ordering, but its microscopic configuration is created by random long range interactions through different closed loops, instead of locally induced magnetic domains.

For low fields the magnetic ordering of large clusters is strongly suppressed, while small clusters keep their ordering. This effect is a consequence of the *small-world* structure which breaks up the global magnetic order, but strongly correlated small clusters survive.

In conclusion, we have shown that random links remove infrared divergences by breaking the inversion symmetry of a complex network. This way, the dimensionality restrictions imposed by the MWT, which assumes translational invariance, are completely removed. Instead, an unconventional second order phase transition emerges due solely to the topology of the system. This topologically induced phase transition displays long

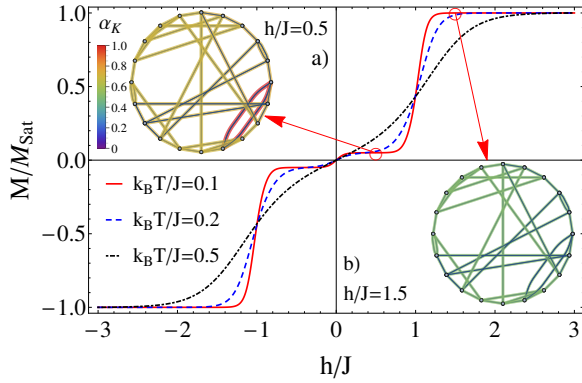


FIG. 5. (color online). m_z vs. h/J loops of an $N = 20$ node 1D Ising ring with one randomly distributed long range link per node, for $k_B T/J = 0.1, 0.2, 0.5$. The correlation length α_K , as a function of cluster size, is given by the colors of the links of the a) and b) insets, for $h/J = 0.5$ and 1.5 , respectively.

TABLE I. Local correlation length α_K , for $K = 2, 7, 8$ and 23 (where $2+7+8+23 = 40$, that is twice the number of nodes) as a function of magnetic field, of a 1D Ising ring system with $N = 20$ nodes and a *small-world* network structure. This network is illustrated as the insets of Fig. (5). The values are normalized to $\alpha_2 = 1$.

h/J	Cluster Decomposition			
	α_2	α_7	α_8	α_{23}
0.5	1.0	0.72	0.74	0.70
1.0	0.67	0.66	0.66	0.66
1.5	0.51	0.51	0.51	0.51

range magnetic order in the absence of external fields through interconnected clusters, as suggested by Graß et al. [29]. Moreover, the *small-world* network topology guarantees that spin correlations are robust against thermal fluctuations.

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